

# SINGULAR DEL PEZZO SURFACES WHOSE UNIVERSAL TORSORS ARE HYPERSURFACES

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ABSTRACT. We classify all singular Del Pezzo surfaces of degree three or greater whose universal torsor is an open subset of a hypersurface in affine space. Equivalently, their Cox ring is a polynomial ring with exactly one relation. For all 20 types with this property, we describe the Cox ring in detail.

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## 1. INTRODUCTION

Universal torsors were introduced by Colliot-Thélène and Sansuc in connection with their studies of the Hasse principle for Del Pezzo surfaces of degrees 3 and 4 [CTS80], [CTS87]. Further developments in this area are due to Salberger [Sal98] and Peyre [Pey98].

Universal torsors have been used successfully by Heath-Brown, Salberger, Browning, and de la Bretèche to prove results in the context of Manin's conjecture [FMT89] concerning the asymptotic of rational points of bounded height for (possibly singular) Del Pezzo surfaces: After the proof of Manin's conjecture for toric varieties, which include several Del Pezzo surfaces, by Batyrev and Tschinkel [BT98], Salberger gave a different proof using universal torsors [Sal98] which was refined in [Bre01]. In specific toric cases, more precise results were proved [HBM99], [Bre98]. For examples of non-toric Del Pezzo surfaces, proofs were found in degree 5 [Bre02], degree 4 [BB], [DT06], and partial results in degree 3 [HB03], [Bro04]. See [Bro05] and [DT06] for an overview of the current progress towards Manin's conjecture for Del Pezzo surfaces.

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In each case, one needs a very precise understanding of the universal torsor, in particular of its equations. In this paper, we focus on universal torsors which have a presentation as Zariski open subsets of hypersurfaces in affine space. A first example, the universal torsor over the unique cubic surface with an isolated singularity of type  $\mathbf{E}_6$ , has been worked out by Hassett and Tschinkel in [HT04]. It lead to the first complete proof of Manin's conjecture for a singular cubic surface in [Der05] and [BBD05].

The universal torsor of a Del Pezzo surface  $S$  is of one of the following three types:

- The universal torsor can be presentable as a Zariski open subset of affine space. This is true exactly when  $S$  is toric.
- Besides this, the case where the universal torsor is an open subset of a hypersurface has been most successfully considered in recent applications.
- In other cases, the universal torsor is more complicated: It can be an open subset of an affine variety that has codimension  $\geq 2$  in affine space.

Universal torsors are closely connected to the homogeneous coordinate rings of Del Pezzo surfaces. These rings were first considered by Cox [Cox95] in the case of toric varieties.

Later, several authors studied these rings, which are now called Cox rings, for other varieties, cf. [HK00], [Muk01], [BH03a], [BH03b], [EKW04], [CT05]. Generators and relations in the Cox rings of smooth Del Pezzo surfaces can be found in [BP04] and [Der06].

Cox proved that the homogeneous coordinate ring is a free polynomial ring exactly for toric varieties. We will see that the question which of the three cases above a universal torsor belongs to is equivalent to the question whether the Cox ring is a polynomial ring with no, one, or more relations.

The following theorem summarizes our results for the various degrees (cf. Table 1):

degree	toric	one relation	more than one relation
9	$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$	–	–
8	smooth, 1 singular	–	–
7	smooth, 1 singular	–	–
6	smooth, 3 singular	2 singular	–
5	2 singular	4 singular	smooth
4	3 singular	7 singular	smooth, 5 singular
3	1 singular	7 singular	smooth, 12 singular

TABLE 1. Relations in Cox rings of Del Pezzo surfaces

**Theorem 1.** *The Cox rings of smooth and singular Del Pezzo surfaces whose degree is at least 3 have the following properties:*

- In degree at least 7, all Del Pezzo surfaces are toric.
- In degree 6, two singular types  $\mathbf{A}_1$  (with three lines),  $\mathbf{A}_2$  have a Cox ring with 7 generators and one relation. The smooth and two singular types  $\mathbf{A}_1$  (with four lines),  $2\mathbf{A}_1$ ,  $\mathbf{A}_2 + \mathbf{A}_1$  are toric.

- In degree 5, four singular types  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$  have a Cox ring with 8 generators and one relation. The smooth surface has a Cox ring with more generators and relations, and two singular types  $2\mathbf{A}_1, \mathbf{A}_2 + \mathbf{A}_1$  are toric.
- In degree 4, seven singular types  $3\mathbf{A}_1, \mathbf{A}_2 + \mathbf{A}_1, \mathbf{A}_3$  (with five lines),  $\mathbf{A}_3 + \mathbf{A}_1, \mathbf{A}_4, \mathbf{D}_4, \mathbf{D}_5$  have a Cox ring with 9 generators and one relation. The smooth surface and five singular types  $\mathbf{A}_1, 2\mathbf{A}_1$  (with eight or nine lines),  $\mathbf{A}_2, \mathbf{A}_3$  (with four lines) have a Cox ring with more generators and relations, and three singular types  $4\mathbf{A}_1, \mathbf{A}_2 + 2\mathbf{A}_1, \mathbf{A}_3 + 2\mathbf{A}_1$  are toric.
- In degree 3, seven singular types  $\mathbf{D}_4, \mathbf{A}_3 + 2\mathbf{A}_1, 2\mathbf{A}_2 + \mathbf{A}_1, \mathbf{A}_4 + \mathbf{A}_1, \mathbf{D}_5, \mathbf{A}_5 + \mathbf{A}_1, \mathbf{E}_6$  have a Cox ring with 10 generators and one relation. The smooth surface and 12 singular types  $\mathbf{A}_1, 2\mathbf{A}_1, \mathbf{A}_2, 3\mathbf{A}_1, \mathbf{A}_2 + \mathbf{A}_1, \mathbf{A}_3, 4\mathbf{A}_1, \mathbf{A}_2 + 2\mathbf{A}_1, \mathbf{A}_3 + \mathbf{A}_1, 2\mathbf{A}_2, \mathbf{A}_4, \mathbf{A}_5$  have a Cox ring with more generators and relations, and the singular type  $3\mathbf{A}_2$  is toric.

In Section 2, we present facts about Del Pezzo surfaces and their universal torsors and Cox rings. We prove results that will be useful in order to show that certain types of Del Pezzo surfaces must have a Cox ring with more than one relation.

In Section 3, we describe the steps which must be taken for each Del Pezzo surface in order to show that there is exactly one relation in its Cox ring, and to calculate it explicitly.

In Sections 4, 5, 6, 7, and 8, we check for every type of degree between 3 and 9 which of the three cases it belongs to. For each type with exactly one relation, we calculate the Cox ring, and we list the data which is most important for its calculation and applications: We give a “nice” split model which is defined over the integers, its singularities, its lines, generators of the Picard group, the effective cone, generators of the Cox ring, the extended Dynkin diagram, and the map from the universal torsor to the Del Pezzo surface.

An overview of the results in each degree can be found at the beginning of these sections in Propositions 8, 9, 11, 12, and 13.

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## 2. UNIVERSAL TORSORS AND COX RINGS

In this section, we summarize basic facts about (possibly singular) Del Pezzo surfaces (see [Man86], [DP80] for more details), their universal torsors, and Cox rings.

Let  $\mathbb{K}$  be an algebraically closed field of characteristic 0. Successive blow-ups of points in  $\mathbb{P}^2$  over  $\mathbb{K}$  have the following effects: The Picard group of  $\mathbb{P}^2$  is free of rank 1. Each blow-up increases the rank of the Picard group by 1. The *intersection form*  $(\cdot, \cdot)$  is a non-degenerate bilinear form on the Picard group. If a prime divisor has self intersection number  $n$ , we call it

an  $(n)$ -curve. If  $n < 0$ , then it is a *negative curve*. The lines on  $\mathbb{P}^2$  are  $(1)$ -curves. More generally, curves of degree  $n$  in  $\mathbb{P}^2$  are  $(n^2)$ -curves. Blowing up a point on a curve decreases its self intersection number by 1 and produces an exceptional divisor which is a  $(-1)$ -curve.

Note that we often use the same notation for curves and their divisor classes; it will be clear from the context what we mean.

Blowing up points in  $\mathbb{P}^2$  creates Del Pezzo surfaces with the following properties:

- A smooth Del Pezzo surface is  $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ , or the blow-up of  $\mathbb{P}^2$  in  $r$  points in *general position* (i.e., no three on one line, no six on a conic and no eight with one of them singular on a cubic curve in  $\mathbb{P}^2$ ), where  $1 \leq r \leq 8$ .
- If the  $r$  blown-up points are not in general, but *almost general position* (for example if we blow up points on the exceptional divisors; “almost general” means that we may blow up successively any point except the ones on  $(-2)$ -curves), we get a surface  $\tilde{S}$  with  $(-2)$ -curves. By contracting them, we obtain a singular Del Pezzo surface  $S$  whose minimal desingularization is  $\tilde{S}$ . The only other singular Del Pezzo surface is the toric Hirzebruch surface  $F_2$ .
- Del Pezzo surfaces are projective Fano varieties of dimension two, i.e., surfaces whose anticanonical divisor is ample.
- The degree of  $S$  is  $d := 9 - r$  in case of the blow-ups of  $\mathbb{P}^2$ , and  $\deg(\mathbb{P}^1 \times \mathbb{P}^1) = \deg(F_2) = 8$ . For  $r \leq 6$ , the anticanonical divisor  $-K_S$  embeds  $S$  into  $\mathbb{P}^d$ .
- The Picard group of  $\tilde{S}$  is free of rank  $r + 1$ .

For the rest of this section, we exclude the cases  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $F_2$ . In the smooth cases, let  $\tilde{S} := S$ . Let  $\mathcal{L}_0, \dots, \mathcal{L}_r$  be a basis of the Picard group of  $\tilde{S}$ . Then the *Cox ring*, or *total coordinate ring*, of  $\tilde{S}$  is defined as

$$\text{Cox}(\tilde{S}) = \bigoplus_{\nu \in \mathbb{Z}^{r+1}} \Gamma(\mathcal{L}^\nu),$$

where we use the notation  $\mathcal{L}^\nu := \mathcal{L}_0^{\otimes \nu_0} \otimes \dots \otimes \mathcal{L}_r^{\otimes \nu_r}$  for  $\nu = (\nu_0, \dots, \nu_r) \in \mathbb{Z}^{r+1}$  and  $\Gamma(\mathcal{L}) := H^0(\tilde{S}, \mathcal{L})$  for  $\mathcal{L} \in \text{Pic}(\tilde{S})$ . It is graded by  $\text{Pic}(\tilde{S})$ .

Let  $\mathcal{L}^\circ := \mathcal{L} \setminus \{\text{zero-section}\}$ . Then the *universal torsor* of  $\tilde{S}$  is

$$\mathcal{T} := \mathcal{L}_0^\circ \times_{\tilde{S}} \dots \times_{\tilde{S}} \mathcal{L}_r^\circ.$$

It is a  $T_{\text{NS}}$ -bundle over  $\tilde{S}$ , where  $T_{\text{NS}} := \text{Hom}(\text{Pic}(\tilde{S}), \mathbb{G}_m) \cong \mathbb{G}_m^{r+1}$  is the Néron-Severi torus.

The universal torsor  $\mathcal{T}$  is an open subset of  $\text{Spec}(\text{Cox}(\tilde{S}))$ .

The following two lemmas give some information on the degrees of  $\text{Cox}(\tilde{S})$  in which its generators can be found.

**Lemma 2.** *If  $\mathcal{L} = [D] \in \text{Pic}(\tilde{S})$  for a prime divisor  $D$  with negative self-intersection number, then  $\dim \Gamma(\mathcal{L}) = 1$ , and every system of generators of  $\text{Cox}(\tilde{S})$  contains a section of degree  $\mathcal{L}$ .*

*The number of generators of  $\text{Cox}(\tilde{S})$  must be at least the number of negative curves.*

*Proof.* Since  $D$  is an effective divisor,  $\Gamma(\mathcal{L})$  is non-trivial. If  $s_1, s_2 \in \Gamma(\mathcal{L})$  were linearly independent, then  $(\mathcal{L}, \mathcal{L})$  could be calculated by looking at the number of intersections of the divisors  $D_1, D_2$  corresponding to the vanishing of  $s_1, s_2$ , which is non-negative.

With this, the second assertion follows from [GM05, Lemma 1], and the last follows immediately from that.  $\square$

However, as the example of the  $\mathbf{E}_6$ -cubic surface [HT04, Section 3] shows, there might be other generators besides sections of negative curves. The following lemma says that we may assume that the degrees of the other generators are nef, without changing their number:

**Lemma 3.** *Let  $E_1, \dots, E_t$  be the negative curves of  $\tilde{S}$ , and let  $\eta_i \in \Gamma(E_i)$  be a non-zero global section. For  $i \in \{1, \dots, s\}$ , let  $\beta_i \in \Gamma(B_i)$  for arbitrary  $B_i \in \text{Pic}(\tilde{S})$ . Let  $A_i$  be the moving part of  $B_i$ .*

*If  $\text{Cox}(\tilde{S})$  is generated by  $\eta_1, \dots, \eta_t, \beta_1, \dots, \beta_s$ , then there are  $\alpha_i \in \Gamma(A_i)$  such that  $\text{Cox}(\tilde{S})$  is generated by  $\eta_1, \dots, \eta_t, \alpha_1, \dots, \alpha_s$ .*

*Proof.* By [HT04, Lemma 3.6], we can write  $B_i = A_i + \sum_{j=1}^t e_j E_j$  with non-negative  $e_j \in \mathbb{Z}$ , where  $A_i$  is the moving part and  $\sum e_j E_j$  is the fixed part of  $B_i$ . Then  $\beta_i = \alpha_i \eta_1^{e_1} \cdots \eta_t^{e_t}$  for some  $\alpha_i \in \Gamma(A_i)$ . Thus, when we replace  $\beta_i$  by  $\alpha_i$ , then the resulting set of sections still generates  $\text{Cox}(\tilde{S})$ .  $\square$

Next, we relate the number of generators of the Cox ring to the type of affine variety in which the universal torsor is an open subset.

**Lemma 4.** *The universal torsor of  $S$  is  $r + 3$ -dimensional.*

- *It is an open subset of affine space (i.e.,  $S$  is toric) if and only if  $\text{Cox}(\tilde{S})$  has exactly  $r + 3$  generators.*
- *It is an open subset of a hypersurface if and only if  $\text{Cox}(\tilde{S})$  has  $r + 4$  generators.*
- *When  $\text{Cox}(\tilde{S})$  has at least  $r + 5$  generators, then there is more than one relation in the Cox ring, and the universal torsor has codimension two or higher in a suitable affine space.*

*Proof.* For the dimension of the universal torsor, we have

$$\dim(\mathcal{T}) = \dim(\tilde{S}) + \dim(\mathbb{G}_m^{r+1}) = r + 3.$$

As the universal torsor is an open subset of  $\text{Spec}(\text{Cox}(\tilde{S}))$ , the Cox ring must be a free polynomial ring with  $r + 3$  generators, or it must have  $r + 4$  generators whose ideal of relations is generated by one equation, or at least  $r + 5$  generators and at least two independent relations.  $\square$

*Remark 5.* For many Del Pezzo surfaces  $S$ , the number of negative curves on its minimal desingularization  $\tilde{S}$  is at least  $r + 5$ , so Lemma 2 and Lemma 4 imply immediately that  $\text{Cox}(\tilde{S})$  has at least two relations.

On the other hand, in many cases where the number of negative curves on  $\tilde{S}$  is at most  $r + 4$ , the surfaces turn out to be toric or to have exactly one relation in  $\text{Cox}(\tilde{S})$ .

However, it can happen that  $\tilde{S}$  has  $r + 4$  or fewer negative curves, but two or more relations (see Proposition 12 for type *viii* of degree 4, and

Proposition 13 for types  $xi$  and  $xvi$  of degree 3). In these cases, some more work must be done to see that the number of relations is actually at least two.

The following lemma gives a criterion which points of  $\mathbb{P}^2$  and subsequently the exceptional divisors can be blown up in order to obtain a toric Del Pezzo surface:

**Lemma 6.** *A Del Pezzo surface is toric if and only if it is  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$  or  $F_2$ , or the blown-up points fulfill the following conditions:*

*There must be three lines in  $\mathbb{P}^2$  forming a triangle such that at most their pairwise intersections and their intersections with exceptional divisors are blown up.*

*In terms of the Dynkin diagram of a blow-up of  $\mathbb{P}^2$ , this means that all the  $(-1)$ - and  $(-2)$ -curves, possibly together with some  $(0)$ - or  $(1)$ -curves, form a “circle”.*

*Proof.* A projective toric surface is described by the one-skeleton of its fan. The one-skeleton of  $\mathbb{P}^2$  consists of three rays corresponding to three lines in  $\mathbb{P}^2$  which intersect pairwise. The one-skeleton of a toric blow-up of  $\mathbb{P}^2$  consists of rays  $\rho_i$  corresponding to these three lines and the exceptional divisors. A ray  $\rho_{i+1}$  corresponding to an exceptional divisor lies between  $\rho_i$  and  $\rho_{i+2}$  if and only if the surface can be obtained from a toric surface by blowing up the intersection of the lines or exceptional divisors corresponding to  $\rho_i$  and  $\rho_{i+2}$ .

The exceptional divisors and lines intersect if and only if the corresponding rays are next to each other in the fan. The exceptional divisors are  $(-1)$ - or  $(-2)$ -curves. The self intersection number of the lines is between  $-2$  and  $1$ , depending on the number of times they have been blown up. No other curves with negative self-intersection can occur as no other lines are blown up more than once. As every ray in the fan has exactly two neighbors, the corresponding Dynkin diagram has the form of a “circle”.  $\square$

*Remark 7.* In general, we denote the intersection behavior of divisor classes which have a generator of the Cox ring as a global section by *extended Dynkin diagrams*: A vertex corresponds to such a divisor class. If two divisor classes have a positive intersection number, we connect their vertices by an appropriate number of lines. Otherwise, their intersection number is zero.

In the toric case, by [Cox95], generators of  $\text{Cox}(\tilde{S})$  are given by one section of each divisor class corresponding to the rays of the fan. In this situation, we denote the extended Dynkin diagram by  $(a_1, \dots, a_{r+3})$ , where  $a_i$  is the self intersection number of the divisor  $D_i$  corresponding to the ray  $\rho_i$ , and  $\rho_i$  is next to  $\rho_{i-1}$  and  $\rho_{i+1}$ , i.e.,  $D_i$  intersects exactly  $D_{i-1}$  and  $D_{i+1}$  (where  $\rho_0 = \rho_{r+3}$ , and  $\rho_1 = \rho_{r+4}$ , and similarly for  $D_i$ ).

### 3. STRATEGY OF THE PROOFS

As we do not want to give all the details of the Cox ring calculation of singular Del Pezzo surfaces in every case where the Cox ring has one relation,

we give an overview of the method, which was developed in [HT04], in this section.

The Cox ring has one generating section for every negative curve. In some cases, the Cox ring is generated by these sections, but this is not always true. In that case, we look for extra generators in nef degrees. When we find  $r + 4$  generators, we have a relation which can be calculated explicitly once we know what  $\phi^*(x_i)$  for the anticanonical embedding  $\phi$  is.

In more detail, we perform the following steps: Let  $S$  be a Del Pezzo surface which is a blow-up of  $\mathbb{P}^2$  in  $r \leq 6$  points and has degree  $9 - r \geq 3$ , given by the vanishing of some homogeneous polynomials for its anticanonical embedding into  $\mathbb{P}^{9-r}$ . Let  $\tilde{S}$  be its minimal desingularization. We apply this method only in the cases where the Cox ring will turn out to have exactly  $r + 4$  generators and one relation. In the other cases, it would fail.

**Find the extended Dynkin diagram of the negative curves.** For this, first we search for the lines on  $S$ , by looking at the equations for  $S$ . Using the classifications of [BW79] for degree 3 and [CT88] for degrees  $\geq 4$ , we know when we are done. Then we determine the singularities, blow them up, and keep track of their intersection behavior with the transforms of the lines. This must be done explicitly for degree 3, and the results can be found in [CT88] for degrees  $\geq 4$ . The extended Dynkin diagram has a vertex for every negative curve, and an edge between two vertices if the corresponding negative curves intersect with intersection number 1. Otherwise, their intersection number is 0. The exceptional divisors have self intersection  $-2$ , and the transforms of the lines on  $S$  have self intersection  $-1$ . (Note that the diagrams can also be recovered from the information given in [AN04, Table 3].)

**Determine a basis of the Picard group.** We need a  $\mathbb{Z}$ -basis for  $\text{Pic}(\tilde{S}) \cong \mathbb{Z}^{r+1}$ : We test for different  $(r + 1)$ -element subsets of the negative curves whether their intersection matrix has determinant  $\pm 1$ . Once we have found such a subset, we call its elements  $E_1, \dots, E_{r+1}$ , and the remaining negative curves are called  $E_{r+2}, \dots, E_t$ . By considering their intersection with the basis, we determine them in terms of this basis. Using the adjunction formula, we calculate the anticanonical divisor  $-K_{\tilde{S}}$ , which is nef since it describes the anticanonical embedding of  $\tilde{S}$ .

**Determine the effective cone and its dual, the nef cone.** In every case, we want to show that the effective cone is generated by the negative curves  $E_1, \dots, E_t$ . Arguing as in [HT04, Prop. 3.5], we only need to check that the cone generated by  $E_1, \dots, E_t$  contains its dual, which is generated by some divisors  $A_1, \dots, A_u$ . This can be done by an explicit calculation using the basis we found in the previous step. It turns out to be true in every case. Therefore,  $A_1, \dots, A_u$  generate the nef cone. By Lemma 2, a generating set of  $\text{Cox}(\tilde{S})$  must contain a non-zero section  $\eta_i \in \Gamma(E_i)$  for every  $i \in \{1, \dots, t\}$ .

**Find generating sections in degrees of generators of the nef cone.** To every  $A_i$ , we can associate a map from  $\tilde{S}$  to projective space. By looking at these maps carefully, we can find extra generators  $\alpha_i \in \Gamma(A_i)$  of the Cox ring. We find generators  $\eta_1, \dots, \eta_t, \alpha_1, \dots, \alpha_s$ , where  $t + s = r + 4$ , which fulfill a certain relation. Assuming that this is a generating set with one

relation, we check that this gives exactly the right number of independent section in each  $\Gamma(A_i)$ , where  $\dim \Gamma(A_i) = \chi(A_i)$  can be calculated using Riemann-Roch and Kawamata-Viehweg as in [HT04, Cor. 1.10].

**Determine  $\phi^*(x_i)$  for the anticanonical embedding.** Especially by considering projections from the singularities and lines on the one hand, and maps to  $\mathbb{P}^2$  corresponding to the fact that  $S$  is a blow-up of  $\mathbb{P}^2$  in  $r$  points on the other hand, and by considering the corresponding transformations of the extended Dynkin diagrams, we find out which projections  $\psi_i : x \mapsto (x_{j_1} : \dots : x_{j_n})$  are given by  $A_i$ , and what  $\psi_i^*(x_{j_l}) \in \Gamma(A_i)$  is. Finally, we combine this information in order to get  $\phi^*(x_i) \in \Gamma(-K_{\tilde{S}})$  for  $\phi : \tilde{S} \rightarrow S \rightarrow \mathbb{P}^{9-r}$ . Using the equations defining  $S$ , we derive a relation  $R$  from this. Note that even though  $S$  might be defined by more than one equation, this gives only one relation in the cases we consider. This gives a map

$$\varrho : \mathbb{K}[\eta_1, \dots, \eta_t, \alpha_1, \dots, \alpha_s] / (R) \rightarrow \text{Cox}(\tilde{S}).$$

**Prove injectivity of  $\varrho$ .** The argument always works in the same way as in [HT04, Theorem 3.8]. We only need to check that  $-K_{\tilde{S}}$  is in the inner of the effective cone, which is true in every case.

**Prove surjectivity of  $\varrho$ .** For this, we follow the proof of [HT04, Prop. 3.9]. We need to check the following two things: The generators of the effective cone  $A_1, \dots, A_u$  are contained in the moving cone corresponding to the  $r+4$  divisors  $E_1, \dots, E_t, A_1, \dots, A_s$  (cf. [HT04, Lemma 3.11]). Furthermore, we check that

$$\sum_{i=1}^t E_i + \sum_{j=1}^s A_j = \deg(R) + A$$

for some nef divisor  $A$  (cf. [HT04, Prop. 3.12]).

We carry out these steps for every Del Pezzo surface which was not known to be toric, and whose number of negative curves is at most  $r+4$ . Except for type *viii* in degree 4 and types *xi*, *xvi* in degree 3, everything works out.

In the next sections, we go through all types of Del Pezzo surfaces of degrees between 3 and 9, and list the following information for each type:

- Equations for the anticanonical embedding are given. We chose “nice” equations in the sense that they are defined over  $\mathbb{Z}$ , all the coefficients are  $\pm 1$ , and the singularities and lines have “nice” forms. For applications, it might be useful to know these equations.
- The equations of the singularities and lines are given, as we need their name  $E_i$  later. Furthermore, they are needed in applications.
- The ordering of the  $E_i$  is chosen in such a way that  $E_1, \dots, E_{r+1}$  are a  $\mathbb{Z}$ -basis of  $\text{Pic}(\tilde{S})$ . We list the other  $E_i$  and  $-K_{\tilde{S}}$  in terms of this basis, as this is needed for the calculations.
- We list generators and the relation of  $\text{Cox}(\tilde{S})$ . Here,  $\eta_i$  is always a non-zero section of  $\Gamma(E_i)$ , which is unique up to a scalar factor. For each extra generator  $\alpha_i$ , we give the degree in terms of our basis of  $\text{Pic}(\tilde{S})$ , specify which projection  $\psi_i : x \mapsto (x_{j_1} : \dots : x_{j_n})$  it defines, and what  $\alpha_i = \psi_i^*(x_{j_l})$  and the corresponding divisor  $A_i$  is. We also describe the image of  $A_i$  under the projection  $\psi : \tilde{S} \rightarrow S$ . We give the equation and the degree of the relation.



- We give the extended Dynkin diagram of the negative curves (where lines are printed as  $E_i$ , and the  $(-2)$ -curves bold as  $\mathbf{E}_i$ ), and we also include the divisors  $A_1, \dots, A_s$  (where an edge means that the intersection number is  $\geq 1$ ). This diagram is probably the most important piece of information, as it is independent of the actual equations of the surface, and most other information is obtained from this diagram in some way.
- For the anticanonical embedding  $\phi$ , we list all  $\phi^*(x_i) \in \Gamma(-K_{\tilde{S}})$ .

#### 4. DEGREE $\geq 7$

**Proposition 8.** *All Del Pezzo surfaces of degree  $\geq 7$  are toric.*

*Proof.* In degree 9, the surface  $\mathbb{P}^2$  is toric.

The surface  $\mathbb{P}^1 \times \mathbb{P}^1$  of degree 8 is toric. When we blow up one point in  $\mathbb{P}^2$ , we obtain a smooth toric surface whose cyclic Dynkin diagram as in Remark 7 is  $(1, 0, -1, 0)$ . Furthermore, by [CT88, Prop. 8.1], the Hirzebruch surface  $F_2$  is the only singular Del Pezzo surface of degree 8, with a singularity of type  $\mathbf{A}_1$ . It is toric, cf. [Ful93, 1.1].

The second point could either be a different point in  $\mathbb{P}^2$  (giving a smooth toric surface of degree 7 with cyclic Dynkin diagram  $(0, -1, -1, -1, 0)$ ) or a point on the exceptional divisor of the first blow-up (making it a  $(-2)$ -curve, giving a singular toric surface with diagram  $(1, 0, -2, -1, -1)$ ).  $\square$

#### 5. DEGREE 6

type	singularities	number of lines	type
0	—	6	toric
<i>i</i>	$\mathbf{A}_1$	4	toric
<i>ii</i>	$\mathbf{A}_1$	3	1 relation
<i>iii</i>	$2\mathbf{A}_1$	2	toric
<i>iv</i>	$\mathbf{A}_2$	2	1 relation
<i>v</i>	$\mathbf{A}_2 + \mathbf{A}_1$	1	toric

TABLE 2. Del Pezzo surfaces of degree 6

**Proposition 9.** *Let  $S$  be a Del Pezzo surface of degree 6, which has one of the types of Table 2.*

- *Types 0, *i*, *iii*, *v* are toric, with cyclic Dynkin diagrams*

$$\begin{aligned}
 &(-1, -1, -1, -1, -1, -1), & (0, -1, -1, -2, -1, -1), \\
 &(0, -2, -1, -2, -1, 0), & (1, 0, -2, -2, -1, -2),
 \end{aligned}$$

*respectively.*

- *Types *ii* and *iv* have Cox rings with seven generators and one relation.*

*Proof.* The classification of singular Del Pezzo surfaces of degree 6 can be found in [CT88, Prop. 8.3]. More directly, we can obtain the types in the following way:

Starting with the two different types of Del Pezzo surfaces of degree 7, it is not hard to find the different possibilities for the third blow-up, and to check that only two of them might lead to non-toric varieties.

The last assertion is checked in the rest of this section, and possible generators and relations are calculated.  $\square$

**Type ii ( $A_1$ ).** It has the following properties:

- It is the intersection of nine quadrics in  $\mathbb{P}^6$ :

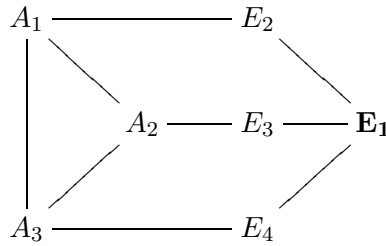
$$\begin{aligned} x_0^2 - x_1x_3 &= x_4x_0 - x_2x_3 = x_2x_0 - x_1x_4 = x_5x_0 - x_2x_4 \\ &= x_5x_1 - x_2^2 = x_5x_3 - x_4^2 = x_0^2 + x_0x_3 + x_6x_4 \\ &= x_0x_1 + x_0^2 + x_6x_2 = x_0x_2 + x_0x_4 + x_6x_5. \end{aligned}$$

- Its singularity  $(0 : 0 : 0 : 0 : 0 : 0 : 1)$  gives the exceptional divisor  $E_1$ , and its lines are  $E_2 = \{x_0 = x_1 = x_2 = x_4 = x_5 = 0\}$ ,  $E_3 = \{x_0 = x_2 = x_3 = x_4 = x_5 = 0\}$ , and  $E_4 = \{x_2 = x_4 = x_5 = x_0 + x_1 = x_0 + x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_4$ , with  $-K_{\tilde{S}} = (3, 2, 2, 2)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_4, \alpha_1, \alpha_2, \alpha_3] / (\eta_2\alpha_1 + \eta_3\alpha_2 + \eta_4\alpha_3),$$

where the degree of the relation is  $(1, 1, 1, 1)$ , and  $\deg(\alpha_1) = A_1 = (1, 0, 1, 1) = \psi^*(\{(0 : 0 : 0 : a^2 : ab : b^2 : 0)\})$ ,  $\deg(\alpha_2) = A_2 = (1, 1, 0, 1) = \psi^*(\{(0 : a^2 : ab : 0 : 0 : b^2 : 0) \mid x_1x_5 - x_2^2 = 0\})$ , and  $\deg(\alpha_3) = A_3 = (1, 1, 1, 0) = \psi^*(\{(a^2 : -a^2 : ab : -a^2 : -ab : -b^2 : 0)\})$ , where  $\psi$  is the projection  $\tilde{S} \rightarrow S$ .

- The extended Dynkin diagram is:



Here, the divisors  $A_1, A_2, A_3$  meet in the point  $(0 : 0 : 0 : 0 : 0 : 1 : 0)$ .

- The anticanonical embedding is given by

$$\begin{aligned} (\phi^*(x_i)) &= (\eta^{(1,1,1,0)}\alpha_1\alpha_2, \eta^{(1,2,0,0)}\alpha_1^2, \eta^{(2,2,1,1)}\alpha_1, \eta^{(1,0,2,0)}\alpha_2^2, \\ &\quad \eta^{(2,1,2,1)}\alpha_2, \eta^{(3,2,2,2)}\alpha_1\alpha_2\alpha_3), \end{aligned}$$

and furthermore,  $\phi^*(-x_0 - x_1) = \eta^{(1,1,0,1)}\alpha_1\alpha_3$ ,  $\phi^*(-x_0 - x_3) = \eta^{(1,0,1,1)}\alpha_2\alpha_3$ ,  $\phi^*(-x_2 - x_4) = \eta^{(2,1,1,2)}\alpha_3$ , and  $\phi^*(2x_0 + x_1 + x_3) = \eta^{(1,0,0,2)}\alpha_3^2$ .

*Remark 10.* Most of this has been calculated by Hassett [Has04, Remark 10]. It is the blow-up of  $\mathbb{P}^2$  in three points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(1 : 1 : 0)$  on a line, where  $E_1$  is the transform of this line,  $E_2, E_3, E_4$  are the exceptional divisors in the three blown-up points, and  $A_1, A_2, A_3$  are the transforms of the lines through  $(0 : 0 : 1)$  and one of the blown-up points.

**Type iv ( $A_2$ ).** This surface has the following properties:

- It is the intersection of the following nine quadrics in  $\mathbb{P}^6$ :

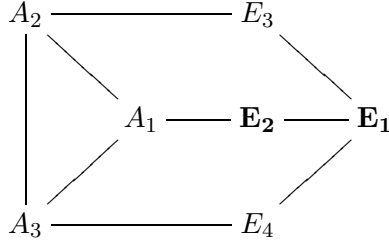
$$\begin{aligned} x_0x_5 - x_3x_4 &= x_0x_6 - x_1x_4 = x_0x_6 - x_2x_3 = x_3x_6 - x_1x_5 \\ &= x_4x_6 - x_2x_5 = x_1x_6 + x_3^2 + x_3x_4 = x_2x_6 + x_3x_4 + x_4^2 \\ &= x_6^2 + x_3x_5 + x_4x_5 = x_1x_2 + x_0x_3 + x_0x_4 = 0. \end{aligned}$$

- Its singularity  $(1 : 0 : 0 : 0 : 0 : 0 : 0)$  gives the exceptional divisors  $E_1, E_2$ , and its lines are  $E_3 = \{x_2 = x_3 = x_4 = x_5 = x_6 = 0\}$  and  $E_4 = \{x_1 = x_3 = x_4 = x_5 = x_6 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_4$ , with  $-K_{\tilde{S}} = (4, 2, 3, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_4, \alpha_1, \alpha_2, \alpha_3] / (\eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3),$$

where the relation is of degree  $(2, 1, 2, 2)$ , and  $\deg(\alpha_1) = A_1 = (1, 0, 1, 1) = \psi^*({(a^2 : 0 : 0 : ab : -ab : b^2 : 0)})$  gives the projection  $\psi_1 : x \mapsto (x_5 : x_6)$  with  $\alpha_1 = \phi^*(x_6)$ ,  $\deg(\alpha_2) = A_2 = (2, 1, 1, 2) = \psi^*({(0 : 0 : a^3 : -a^2b : b^3 : -ab^2)})$  gives the projection  $\psi_2 : x \mapsto (x_3 : x_5 : x_6)$  with  $\alpha_2 = \psi_2^*(x_3)$ , and  $\deg(\alpha_3) = A_3 = (2, 1, 2, 1) = \psi^*({(0 : a^3 : 0 : -a^2b : 0 : b^3 : -ab^2)})$  gives the projection  $\psi_3 : x \mapsto (x_4 : x_5 : x_6)$  with  $\alpha_3 = \psi_3^*(x_4)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, A_3$  meet in  $(0 : 0 : 0 : 0 : 0 : 1 : 0)$ .

- The anticanonical embedding is given by

$$\begin{aligned} (\phi^*(x_i)) &= (\alpha_2\alpha_3, \eta^{(1,1,1,0)}\alpha_1\alpha_2, \eta^{(1,1,0,1)}\alpha_1\alpha_3, \\ &\quad \eta^{(2,1,2,1)}\alpha_2, \eta^{(2,1,1,2)}\alpha_3, \eta^{(4,2,3,3)}, \eta^{(3,2,2,2)}\alpha_1), \end{aligned}$$

and furthermore,  $\phi^*(-x_1 - x_2) = \eta^{(1,2,0,0)}\alpha_1^3$  and  $\phi^*(-x_3 - x_4) = \eta^{(2,2,1,1)}\alpha_1^2$ .

## 6. DEGREE 5

By [CT88, Prop. 8.4], Table 3 lists all types of Del Pezzo surfaces of degree 5.

type	singularities	number of lines	type
0	—	10	$\geq 2$ relations
<i>i</i>	$\mathbf{A}_1$	7	1 relation
<i>ii</i>	$2\mathbf{A}_1$	5	toric
<i>iii</i>	$\mathbf{A}_2$	4	1 relation
<i>iv</i>	$\mathbf{A}_2 + \mathbf{A}_1$	3	toric
<i>v</i>	$\mathbf{A}_3$	2	1 relation
<i>vi</i>	$\mathbf{A}_4$	1	1 relation

TABLE 3. Del Pezzo surfaces of degree 5

**Proposition 11.** *The Del Pezzo surfaces of degree 5 of Table 3 can be divided into the following groups:*

- Type 0 has a Cox ring with 10 generators and five relations.
- Types *ii* and *iv* are toric with cyclic Dynkin diagrams

$$(-1, -1, -1, -1, -2, -1, -2), \quad (0, -1, -1, -2, -2, -1, -2).$$

- Types *i, iii, v, vi* have a Cox ring with 9 generators and one relation.

*Proof.* Type 0 has 10 negative curves. By Lemma 2,  $\text{Cox}(\tilde{S})$  has at least 10 generators. By Lemma 4, this implies that there is more than one relation in  $\text{Cox}(\tilde{S})$ . More information on this surface can be found in [Sko93], [Has04, Section 2.2], [Bre02].

By considering the different possibilities to blow up one of the toric types of degree 6 in such a way that the resulting surface is again toric (see Lemma 6), we get exactly types *ii* and *iv*.

For the other types, we calculate the Cox ring in what follows, and we see that each of them has exactly one relation.  $\square$

**Type i ( $\mathbf{A}_1$ ).** The surface of type *i* has the properties:

- It is the intersection of the following five quadrics in  $\mathbb{P}^5$ :

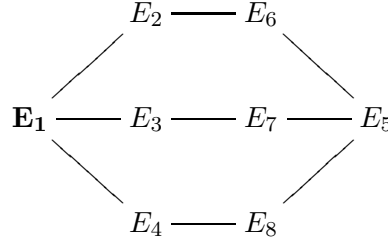
$$\begin{aligned} x_0x_4 - x_1x_2 &= x_0x_5 - x_1x_3 = x_2x_5 - x_3x_4 \\ &= x_1x_2 + x_1x_3 + x_2x_3 = x_1x_4 + x_1x_5 + x_2x_5 = 0. \end{aligned}$$

- The singularity in  $p = (1 : 0 : 0 : 0 : 0 : 0)$  gives an exceptional divisor  $E_1$ , the seven lines on  $S$  are  $E_2 = \{x_1 = x_2 = x_4 = x_5 = 0\}$ ,  $E_3 = \{x_1 = x_3 = x_4 = x_5 = 0\}$ ,  $E_4 = \{x_2 = x_3 = x_4 = x_5 = 0\}$ ,  $E_5 = \{x_0 = x_1 = x_2 = x_3 = 0\}$ ,  $E_6 = \{x_0 = x_1 = x_2 = x_4 = 0\}$ ,  $E_7 = \{x_0 = x_1 = x_3 = x_5 = 0\}$ ,  $E_8 = \{x_0 = x_2 = x_3 = x_4 + x_5 = 0\}$ .
- The classes  $E_1, \dots, E_5$  are a basis of  $\text{Pic}(\tilde{S})$ , with  $E_6 = (1, 0, 1, 1, -1)$ ,  $E_7 = (1, 1, 0, 1, -1)$ ,  $E_8 = (1, 1, 1, 0, -1)$ , and  $-K_{\tilde{S}} = (3, 2, 2, 2, -1)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_8] / (\eta_2\eta_6 + \eta_3\eta_7 + \eta_4\eta_8),$$

and the relation is of degree  $(1, 1, 1, 1, -1)$ .

- The extended Dynkin diagram is:



- The anticanonical embedding  $S \rightarrow \mathbb{P}^5$  is given by

$$(\phi^*(x_i)) = (\eta^{(0,0,0,0,2)}\eta_6\eta_7\eta_8, \eta^{(1,1,1,0,1)}\eta_6\eta_7, \eta^{(1,1,0,1,1)}\eta_6\eta_8, \\ \eta^{(1,0,1,1,1)}\eta_7\eta_8, \eta^{(2,2,1,1,0)}\eta_6, \eta^{(2,1,2,1,0)}\eta_7).$$

In addition,  $\phi^*(-x_4 - x_5) = \eta^{(2,1,1,2,0)}\eta_8$ ,  $\phi^*(-x_1 - x_2) = \eta^{(1,2,0,0,1)}\eta_6^2$ ,  $\phi^*(-x_1 - x_3) = \eta^{(1,0,2,0,1)}\eta_7^2$ ,  $\phi^*(-x_2 - x_3) = \eta^{(1,0,0,2,1)}\eta_8^2$ .

**Type iii ( $\mathbf{A}_2$ ).** The surface of type *iii* has the properties:

- It is the intersection of the following five quadrics in  $\mathbb{P}^5$ :

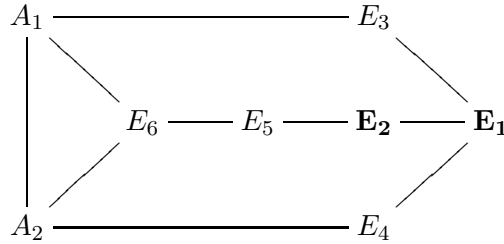
$$x_0x_2 - x_1x_5 = x_0x_2 - x_3x_4 = x_0x_3 + x_1^2 + x_1x_4 \\ = x_0x_5 + x_1x_4 + x_4^2 = x_3x_5 + x_1x_2 + x_2x_4 = 0.$$

- The singularity in  $p = (0 : 0 : 1 : 0 : 0 : 0)$  gives the exceptional divisors  $E_1, E_2$ , and the lines are  $E_3 = \{x_0 = x_1 = x_3 = x_4 = 0\}$ ,  $E_4 = \{x_0 = x_1 = x_4 = x_5 = 0\}$ ,  $E_5 = \{x_0 = x_3 = x_5 = x_1 + x_4 = 0\}$ , and  $E_6 = \{x_2 = x_3 = x_5 = x_1 + x_4 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_5$ , with  $E_6 = (1, 0, 1, 1, -1)$  and  $-K_{\tilde{S}} = (3, 2, 2, 2, 1)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_6, \alpha_1, \alpha_2] / (\eta_2\eta_5^2\eta_6 + \eta_3\alpha_1 + \eta_4\alpha_2),$$

where  $\deg(\alpha_1) = A_1 = (1, 1, 0, 1, 1) = \psi^*(\{(a^2 : 0 : ab : 0 : 0 : -b^2)\})$ ,  $\deg(\alpha_2) = A_2 = (1, 1, 1, 0, 1) = \psi^*(\{(a^2 : ab : 0 : -b^2 : 0 : 0)\})$ , and the degree of the relation is  $(1, 1, 1, 1, 1)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, E_6$  meet in  $(1 : 0 : 0 : 0 : 0 : 0)$ .

- The anticanonical embedding  $\phi : S \rightarrow \mathbb{P}^5$  is given by

$$(\phi^*(x_i)) = (\eta^{(3,2,2,2,1)}, \eta^{(2,1,2,1,0)}\alpha_1, \eta_6\alpha_1\alpha_2, \\ \eta^{(1,1,1,0,1)}\eta_6\alpha_1, \eta^{(2,1,1,2,0)}\alpha_2, \eta^{(1,1,0,1,1)}\eta_6\alpha_2),$$

and furthermore,  $\phi^*(-x_1 - x_4) = \eta^{(2,2,1,1,2)}\eta_6$  and  $\phi^*(-x_3 - x_5) = \eta^{(1,2,0,0,3)}\eta_6^2$ .

**Type v ( $\mathbf{A}_3$ ).** The surface of type  $v$  has the following properties:

- It is the intersection of the following five quadrics in  $\mathbb{P}^5$ :

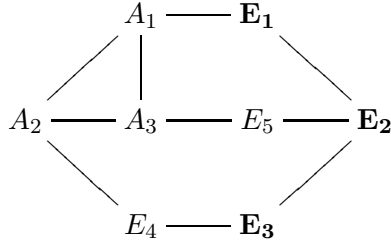
$$\begin{aligned} x_0x_2 - x_1^2 &= x_0x_3 - x_1x_4 = x_2x_4 - x_1x_3 \\ &= x_2x_4 + x_4^2 + x_0x_5 = x_2x_3 + x_3x_4 + x_1x_5 = 0. \end{aligned}$$

- The singularity is in  $p = (0 : 0 : 0 : 0 : 0 : 1)$ , giving three exceptional divisors  $E_1, E_2, E_3$ . The lines  $E_4 = \{x_0 = x_1 = x_2 = x_4 = 0\}$  and  $E_5 = \{x_0 = x_1 = x_3 = x_4 = 0\}$  intersect in  $p$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_5$ , with  $-K_{\tilde{S}} = (2, 4, 3, 2, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_5, \alpha_1, \alpha_2, \alpha_3] / (\eta_1\alpha_1^2 + \eta_3\eta_4^2\alpha_2 + \eta_5\alpha_3),$$

where  $\deg(\alpha_1) = A_1 = (0, 1, 1, 1, 1) = \psi^*(\{(a^2 : 0 : 0 : 0 : ab : -b^2)\})$ ,  $\deg(\alpha_2) = A_2 = (1, 2, 1, 0, 2) = \psi^*(\{(a^2 : ab : b^2 : 0 : 0 : 0)\})$ ,  $\deg(\alpha_3) = A_3 = (1, 2, 2, 2, 1) = \psi^*(\{(a^3 : a^2b : ab^2 : -b^3 : -ab^2 : 0)\})$ , and the degree of the relation is  $(1, 2, 2, 2, 2)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, A_3$  meet in  $(1 : 0 : 0 : 0 : 0 : 0)$ .

- The anticanonical embedding  $\phi : S \rightarrow \mathbb{P}^5$  is given by

$$\begin{aligned} (\phi^*(x_i)) &= (\eta^{(2,4,3,2,3)}, \eta^{(2,3,2,1,2)}\alpha_1, \eta^{(2,2,1,0,1)}\alpha_1^2, \\ &\quad \eta^{(1,1,1,1,0)}\alpha_1\alpha_2, \eta^{(1,2,2,2,1)}\alpha_2, \alpha_2\alpha_3), \end{aligned}$$

and furthermore,  $\phi^*(-x_2 - x_4) = \eta^{(1,2,1,0,2)}\alpha_3$ .

**Type vi ( $\mathbf{A}_4$ ).** The surface of type  $vi$  has the following properties:

- It is given by the intersection of the following five quadrics in  $\mathbb{P}^5$ :

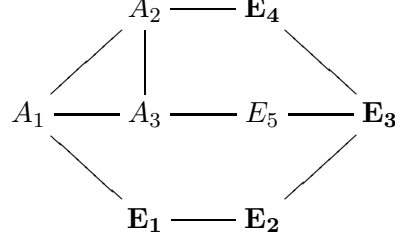
$$\begin{aligned} x_0x_2 - x_1^2 &= x_0x_3 - x_1x_4 = x_2x_4 - x_1x_3 \\ &= x_1x_2 + x_4^2 + x_0x_5 = x_2^2 + x_3x_4 + x_1x_5 = 0. \end{aligned}$$

- The singularity is  $p = (0 : 0 : 0 : 0 : 0 : 1)$ , giving four exceptional divisors  $E_1, \dots, E_4$  in  $\text{Pic}(\tilde{S})$ , and the line is  $E_5 = \{x_0 = x_1 = x_2 = x_4 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_5$ , with  $-K_{\tilde{S}} = (2, 4, 6, 3, 5)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_5, \alpha_1, \alpha_2, \alpha_3] / (\eta_1^2\eta_2\alpha_1^3 + \eta_4\alpha_2^2 + \eta_5\alpha_3),$$

where  $\deg(\alpha_1) = A_1 = (0, 1, 2, 1, 2) = \psi^*(\{(a^2 : 0 : 0 : 0 : ab : -b^2)\})$ ,  $\deg(\alpha_2) = A_2 = (1, 2, 3, 1, 3) = \psi^*(\{(a^3 : a^2b : ab^2 : 0 : 0 : -b^3)\})$ ,  $\deg(\alpha_3) = A_3 = (2, 4, 6, 3, 5) = -K_{\tilde{S}} = \psi^*(\{(a^5 : -a^3b^2 : ab^4 : -b^5 : a^2b^3 : 0)\})$ . The degree of the relation is  $(2, 4, 6, 3, 6)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, A_3$  meet in  $(1 : 0 : 0 : 0 : 0 : 0)$ .

- The anticanonical embedding  $\phi : S \rightarrow \mathbb{P}^5$  is given by

$$(\phi^*(x_i)) = (\eta^{(2,4,6,3,5)}, \eta^{(2,3,4,2,3)}\alpha_1, \eta^{(2,2,2,1,1)}\alpha_1^2, \eta^{(1,1,1,1,0)}\alpha_1\alpha_2, \eta^{(1,2,3,2,2)}\alpha_2, \alpha_3).$$

## 7. DEGREE 4

By classical results, which can be found in [HP52, Book IV, §XIII.11], every Del Pezzo surface of degree 4 is the intersection of two quadrics in  $\mathbb{P}^4$ , given by symmetric  $5 \times 5$  matrices  $A, B$ , where  $A$  can be assumed to be non-singular. Besides the smooth quartic Del Pezzo surface, there are 15 singular types, which can be distinguished by the *Segre symbol* (describing the structure of the Jordan form) of  $A^{-1}B$ . The extended Dynkin diagrams of their lines and exceptional divisors can be found in [CT88, Prop. 6.1].

type	Segre symbol	singularities	number of lines	type
0	$(1, 1, 1, 1, 1)$	—	16	$\geq 2$ relations
<i>i</i>	$(2, 1, 1, 1, 1)$	$\mathbf{A}_1$	12	$\geq 2$ relations
<i>ii</i>	$(2, 2, 1, 1, 1)$	$2\mathbf{A}_1$	9	$\geq 2$ relations
<i>iii</i>	$((1, 1), 1, 1, 1, 1)$	$2\mathbf{A}_1$	8	$\geq 2$ relations
<i>iv</i>	$(3, 1, 1, 1, 1)$	$\mathbf{A}_2$	8	$\geq 2$ relations
<i>v</i>	$((1, 1), 2, 1, 1, 1)$	$3\mathbf{A}_1$	6	1 relation
<i>vi</i>	$(3, 2, 1, 1, 1)$	$\mathbf{A}_2 + \mathbf{A}_1$	6	1 relation
<i>vii</i>	$(4, 1, 1, 1, 1)$	$\mathbf{A}_3$	5	1 relation
<i>viii</i>	$((2, 1), 1, 1, 1, 1)$	$\mathbf{A}_3$	4	$\geq 2$ relations
<i>ix</i>	$((1, 1), (1, 1), 1, 1, 1)$	$4\mathbf{A}_1$	4	toric
<i>x</i>	$((1, 1), 3, 1, 1, 1)$	$\mathbf{A}_2 + 2\mathbf{A}_1$	4	toric
<i>xi</i>	$((2, 1), 2, 1, 1, 1)$	$\mathbf{A}_3 + \mathbf{A}_1$	3	1 relation
<i>xii</i>	$(5, 1, 1, 1, 1)$	$\mathbf{A}_4$	3	1 relation
<i>xiii</i>	$((3, 1), 1, 1, 1, 1)$	$\mathbf{D}_4$	2	1 relation
<i>xiv</i>	$((2, 1), (1, 1), 1, 1, 1)$	$\mathbf{A}_3 + 2\mathbf{A}_1$	2	toric
<i>xv</i>	$((4, 1), 1, 1, 1, 1)$	$\mathbf{D}_5$	1	1 relation

TABLE 4. Del Pezzo surfaces of degree 4

- Types  $ix, x, xiv$  are toric, with cyclic Dynkin diagrams

- The Cox rings of types  $v, vi, vii, xi, xii, xiii, xv$  have 9 generators and one relation.

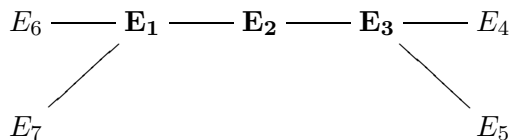
- The Cox rings of types 0, i, ii, iii, iv, viii have at least 10 generators and at least two independent relations.

*Proof.* Blowing down a  $(-1)$ -curve on  $\tilde{S}$  for a toric Del Pezzo surface  $S$  of degree 4 gives a surface  $\tilde{S}'$  which is the minimal desingularization of a toric Del Pezzo surface  $S'$  of degree 5. It is easy to check that the different possibilities to blow up types *ii* and *iv* of degree 5 in such a way that the result is toric give exactly three different toric surfaces in degree 4. Using the results of [CT88, Prop. 6.1], they are identified as types *ix*, *x*, *xiv*.

A singularity of type  $A_j$  or  $D_j$  gives exactly  $j$  exceptional divisors on  $\tilde{S}$ . The lines and exceptional divisors on  $\tilde{S}$  are exactly the negative curves. For types 0,  $i$ ,  $ii$ ,  $iii$ ,  $iv$ , their total number is greater than  $r + 4 = 9$ , so  $\text{Cox}(\tilde{S})$  has more than 9 generators by Lemma 2, and by Lemma 4, there must be more than one relation.

For type *viii*, this is not as obvious because it has only 7 negative curves. We can derive the following information from the extended Dynkin diagram of negative curves given in [CT88, Prop 6.1]:

- The singularity gives the exceptional divisors  $E_1, E_2, E_3$ , and they intersect the four lines  $E_4, \dots, E_7$  in the following way:



- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_6$ , with  $E_7 = (-1, 0, 1, 1, 1, -1)$  and  $-K_{\tilde{S}} = (1, 2, 3, 2, 2, 0)$ .
- Four of the ten generators of the nef cone are  $B_1 = (1, 1, 1, 0, 1, 1)$ ,  $B_2 = (1, 1, 1, 1, 0, 1)$ ,  $B_3 = (0, 1, 2, 1, 2, -1)$ ,  $B_4 = (0, 1, 2, 2, 1, -1)$ , and  $\dim(\Gamma(B_i)) = \chi(B_i) = 2$  for  $i \in \{1, \dots, 4\}$ .

The subring generated by non-zero section  $\eta_j \in \Gamma(E_j)$  for  $j \in \{1, \dots, 7\}$  does not contain two linearly independent sections in any of these degrees  $B_i$ .

Consider a minimal set of generators of  $\text{Cox}(\tilde{S})$ . By Lemma 3, we can assume that it has the form  $\eta_1, \dots, \eta_7, \alpha_1, \dots, \alpha_s$  with all  $A_l := \deg(\alpha_l)$  nef. For all  $i \in \{1, \dots, 4\}$ , it contains two linearly independent sections of degree  $B_l$ , so it contain a section  $\beta_l$  which is a scalar multiple of  $\eta_1^{e_1} \cdots \eta_7^{e_7} \alpha_1^{a_1} \cdots \alpha_s^{a_s}$  where  $e_j, a_l \geq 0$  and not all  $a_l$  are zero. Considering their degrees, this means that  $\sum_{l=1}^s a_l A_l = B_i - \sum_{j=1}^7 e_j E_j$ , and this degree is nef. However, we calculate directly that the intersection of the nef cone with the negative of the effective cone translated by  $B_i$  contains only  $(0, 0, 0, 0, 0, 0)$  and  $B_i$ . Therefore,  $B_i = A_n$  for some  $n \in \{1, \dots, s\}$ , and all  $e_j, a_l$  are zero except



$a_n = 1$ . As this is true for all  $i \in \{1, \dots, 4\}$ , the Cox ring must have at least 11 generators.

For the other types, the total number of negative curves is at most  $r+3 = 9$  by the extended Dynkin diagrams of [CT88], and they are not toric. In the following, we calculate their Cox ring, which will show that it has exactly 9 generators and one relation.  $\square$

**Type v ( $3A_1$ ).** Type  $v$ , which has been considered in [Bro05], has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

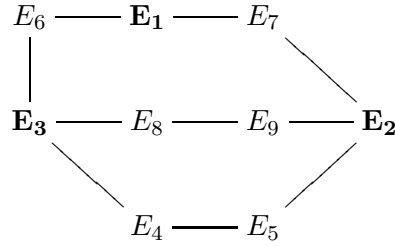
$$x_0x_1 - x_2^2 = x_1x_2 + x_2^2 + x_3x_4 = 0.$$

- Its three singularities  $p_1 = (1 : 0 : 0 : 0 : 0)$ ,  $p_2 = (0 : 0 : 0 : 0 : 1)$ ,  $p_3 = (0 : 0 : 0 : 0 : 1)$  give exceptional divisors  $E_1, E_2, E_3$ , respectively. The six lines are  $E_4 = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_5 = \{x_0 = x_2 = x_4 = 0\}$ ,  $E_6 = \{x_1 = x_2 = x_3 = 0\}$ ,  $E_7 = \{x_1 = x_2 = x_4 = 0\}$ ,  $E_8 = \{x_0 + x_2 = x_1 + x_2 = x_3 = 0\}$ ,  $E_9 = \{x_0 + x_2 = x_1 + x_2 = x_4 = 0\}$
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , and  $E_7 = (-1, 0, 0, 1, 1, -1)$ ,  $E_8 = (-1, 1, -1, 1, 2, -2)$ ,  $E_9 = (1, -1, 1, 0, -1, 2)$ , and  $-K_{\tilde{S}} = (0, 1, 1, 2, 2, 0)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_9]/(\eta_4\eta_5 + \eta_1\eta_6\eta_7 + \eta_8\eta_9),$$

where the relation is of degree  $(0, 0, 0, 1, 1, 0)$ .

- The extended Dynkin diagram is:



- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(0,1,1,2,2,0)}, \eta^{(2,1,1,0,0,2)}\eta_7^2, \eta^{(1,1,1,1,1,1)}\eta_7, \\ \eta^{(1,0,2,1,0,2)}\eta_8, \eta^{(1,2,0,0,1,0)}\eta_7^2\eta_9),$$

and furthermore,  $\phi^*(-x_0 - x_2) = \eta^{(0,1,1,1,1,0)}\eta_8\eta_9$ ,  $\phi^*(-x_1 - x_2) = \eta^{(1,1,1,0,0,1)}\eta_7\eta_8\eta_9$ ,  $\phi^*((x_0 + x_2) + (x_1 + x_2)) = \eta^{(0,1,1,0,0,0)}\eta_8^2\eta_9^2$ .

**Type vi ( $A_2 + A_1$ ).** Type  $vi$  has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

$$x_0x_1 - x_2x_3 = x_1x_2 + x_2x_4 + x_3x_4 = 0.$$

- Its singularity  $p_1 = (1 : 0 : 0 : 0 : 0)$  gives the exceptional divisors  $E_1, E_2$ , and  $p_2 = (0 : 0 : 0 : 0 : 1)$  gives  $E_3$ . The six lines are  $E_4 = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_5 = \{x_0 = x_2 = x_4 = 0\}$ ,  $E_6 = \{x_1 = x_3 =$

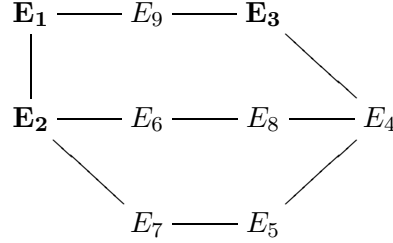
$x_4 = 0\}$ ,  $E_7 = \{x_1 = x_2 = x_4 = 0\}$ ,  $E_8 = \{x_0 = x_3 = x_1 + x_4 = 0\}$ ,  $E_9 = \{x_1 = x_2 = x_3 = 0\}$ .

- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_6$ , with  $E_7 = (-1, -2, 1, 2, 1, -2)$ ,  $E_8 = (-1, -2, 1, 2, 2, -3)$ ,  $E_9 = (-1, -1, 0, 1, 1, -1)$ , and  $-K_{\tilde{S}} = (-1, -2, 2, 4, 3, -3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_9]/(\eta_5\eta_7 + \eta_1\eta_3\eta_9^2 + \eta_6\eta_8),$$

where the relation is of degree  $(-1, -2, 1, 2, 2, -2)$ .

- The extended Dynkin diagram is:



- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(0,0,1,2,1,0)}\eta_8, \eta^{(2,2,1,0,0,1)}\eta_7\eta_9^2, \eta^{(1,1,1,1,1,0)}\eta_7\eta_9, \\ \eta^{(1,1,1,1,0,1)}\eta_8\eta_9, \eta^{(1,2,0,0,1,1)}\eta_7^2),$$

and furthermore,  $\phi^*(-x_1 - x_4) = \eta^{(1,2,0,0,0,2)}\eta_7\eta_8$  and  $\phi^*(-x_2 - x_3) = \eta^{(2,1,2,1,0,0)}\eta_9^3$

**Type vii (A<sub>3</sub>).** Type *vii* has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

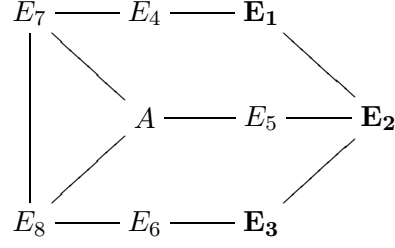
$$x_0x_1 - x_2x_3 = x_2x_4 + x_0x_3 + x_1x_3 = 0.$$

- Its singularity  $(0 : 0 : 0 : 0 : 1)$  gives the exceptional divisors  $E_1, E_2, E_3$ , and the five lines are  $E_4 = \{x_1 = x_2 = x_3 = 0\}$ ,  $E_5 = \{x_0 = x_1 = x_2 = 0\}$ ,  $E_6 = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_7 = \{x_1 = x_3 = x_4 = 0\}$ ,  $E_8 = \{x_0 = x_3 = x_4 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , with  $E_7 = (0, 1, 1, -1, 1, 1)$ ,  $E_8 = (1, 1, 0, 1, 1, -1)$ , and  $-K_{\tilde{S}} = (2, 3, 2, 1, 2, 1)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_8, \alpha]/(\eta_5\alpha + \eta_1\eta_4^2\eta_7 + \eta_3\eta_6^2\eta_8),$$

where the relation is of degree  $(1, 1, 1, 1, 1, 1)$ , and  $\deg(\alpha) = A = (1, 1, 1, 1, 0, 1) = \psi^*({(ab : -ab : b^2 : a^2 : 0)})$  gives the projection  $\psi_1 : x \mapsto (x_3 : x_4)$  with  $\alpha = \psi_1^*(x_4)$ .

- The extended Dynkin diagram is:



Here,  $A, E_7, E_8$  meet in  $(0 : 0 : 1 : 0 : 0)$ .

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(2,2,1,2,1,0)}\eta_7, \eta^{(1,2,2,0,1,2)}\eta_8, \eta^{(2,3,2,1,2,1)}\eta_7\eta_8, \eta^{(1,1,1,1,0,1)}\eta_7\eta_8\alpha),$$

and furthermore,  $\phi^*(-x_0 - x_1) = \eta^{(1,2,1,0,2,0)}\alpha$

**Type xi ( $\mathbf{A}_3 + \mathbf{A}_1$ ).** Type  $xi$  has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

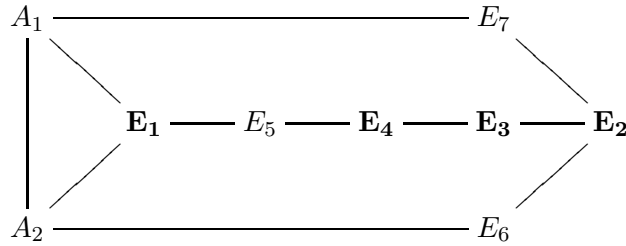
$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0.$$

- The singularity  $p_1 = (0 : 1 : 0 : 0 : 0)$  gives an exceptional divisor  $E_1$ , and  $p_2 = (0 : 0 : 0 : 0 : 1)$  gives  $E_2, E_3, E_4$ . The three lines are  $E_5 = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_6 = \{x_0 = x_1 = x_2 = 0\}$ ,  $E_7 = \{x_1 = x_2 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , with  $E_7 = (1, -1, 0, 1, 2, -1)$  and  $-K_{\tilde{S}} = (2, 1, 2, 3, 4, 0)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2] / (\eta_6\alpha_2 + \eta_7\alpha_1 + \eta_1\eta_3\eta_4^2\eta_5^3),$$

where the relation is of degree  $(1, 0, 1, 2, 3, 0)$ ,  $\deg(\alpha_1) = A_1 = (0, 1, 1, 1, 1, 1) = \psi^*(\{(a^2 : -b^2 : ab : 0 : 0)\})$  gives the projection  $\psi_1 : x \mapsto (x_2 : x_3)$  with  $\alpha_1 = \psi_1^*(x_3)$ , and  $\deg(\alpha_2) = A_2 = (1, 0, 1, 2, 3, -1) = \psi^*(\{(0 : a^2 : ab : -b^2 : 0)\})$  gives the projection  $\psi_2 : x \mapsto (x_0 : x_2)$  with  $\alpha_2 = \psi_2^*(x_0)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, E_1$  meet in one point.

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(1,1,1,1,1,1)}\alpha_2, \eta^{(0,3,2,1,0,2)}\eta_7^2, \eta^{(1,2,2,2,2,1)}\eta_7, \eta^{(1,1,1,1,1,0)}\eta_7\alpha_1, \eta^{(1,0,0,0,0,0)}\alpha_1\alpha_2),$$

and furthermore,  $\phi^*(-x_0 - x_3) = \eta^{(2,1,2,3,4,0)}$ .

**Type xii (A<sub>4</sub>).** Type *xii* has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

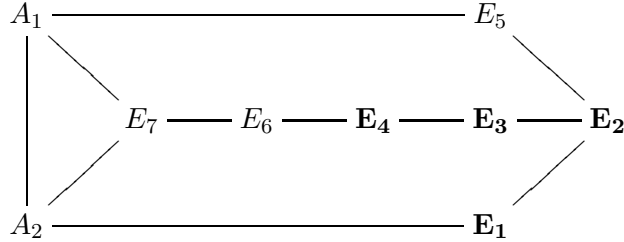
$$x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0.$$

- The singularity  $p = (0 : 0 : 0 : 0 : 1)$  gives the exceptional divisors  $E_1, \dots, E_4$ , and the three lines are  $E_5 = \{x_0 = x_2 = x_3 = 0\}$ ,  $E_6 = \{x_0 = x_1 = x_3 = 0\}$ , and  $E_7 = \{x_1 = x_3 = x_4 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , with  $E_7 = (1, 2, 1, 0, 2, -1)$ , and  $-K_{\tilde{S}} = (2, 4, 3, 2, 3, 1)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2] / (\eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7),$$

where the degree of the relation is  $(1, 2, 2, 2, 2, 2)$ , and  $\deg(\alpha_1) = A_1 = (1, 2, 2, 2, 1, 2) = \psi^*(\{(-b^3 : a^3 : -ab^2 : a^2b : 0)\})$  gives the projection  $\psi_1$  from  $E_7$ , with  $\alpha_1 = \psi_1^*(x_4)$ , and  $\deg(\alpha_2) = A_2 = (0, 1, 1, 1, 1, 1) = \psi^*(\{(a^2 : 0 : 0 : -ab : b^2)\})$  gives the projection  $\psi_2 : x \mapsto (x_1 : x_3)$  with  $\alpha_2 = \psi_2^*(x_1)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, E_7$  meet in  $(1 : 0 : 0 : 0 : 0)$ .

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(2,4,3,2,3,1)}, \eta^{(1,1,1,1,0,1)}\eta_7\alpha_2, \eta^{(2,3,2,1,2,0)}\alpha_2, \eta^{(1,2,2,2,1,2)}\eta_7, \eta_7\alpha_1).$$

**Type xiii (D<sub>4</sub>).** Type *xiii*, which was considered in more detail in [DT06] (see [BB05] for a non-split form), has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

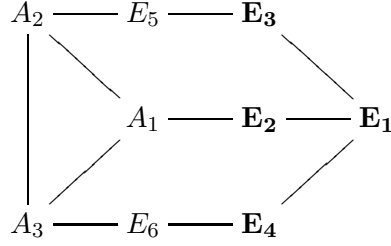
$$x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0.$$

- The singularity  $p = (0 : 0 : 0 : 0 : 1)$  gives four exceptional divisor  $E_1, \dots, E_4$ , and its lines are  $E_5 = \{x_0 = x_1 = x_2 = 0\}$  and  $E_6 = \{x_1 = x_2 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , and  $-K_{\tilde{S}} = (4, 2, 3, 3, 2, 2)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_6, \alpha_1, \alpha_2, \alpha_3] / (\eta_3\eta_5^2\alpha_2 + \eta_4\eta_6^2\alpha_3 + \eta_2\alpha_1^2),$$

where the degree of the relation is  $(2, 1, 2, 2, 2, 2)$ ,  $\deg(\alpha_1) = A_1 = (1, 0, 1, 1, 1, 1) = \psi^*(\{(ab : b^2 : 0 : -ab : -a^2)\})$  gives the projection  $\psi_1 : x \mapsto (x_1 : x_2)$  with  $\alpha_1 = \psi_1^*(x_2)$ ,  $\deg(\alpha_2) = A_2 = (2, 1, 1, 2, 0, 2) = \psi^*(\{(0 : a^2 : ab : -b^2 : 0)\})$  gives the projection  $\psi_2 : x \mapsto (x_0 : x_1)$  with  $\alpha_2 = \psi_2^*(x_0)$ , and  $\deg(\alpha_3) = A_3 = (2, 1, 2, 1, 2, 0) = \psi^*(\{(a^2 : -b^2 : ab : 0 : 0)\})$  gives the projection  $\psi_3 : x \mapsto (x_1 : x_3)$  with  $\alpha_3 = \psi_3^*(x_3)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, A_3$  intersect in  $(0 : 1 : 0 : 0 : 0)$ .

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(2,1,2,1,2,0)}\alpha_2, \eta^{(4,2,3,3,2,2)}\alpha_1, \eta^{(3,2,2,2,1,1)}\alpha_1, \eta^{(2,1,1,2,0,2)}\alpha_3, \alpha_2\alpha_3).$$

Furthermore,  $\phi^*(-x_0 - x_3) = \eta^{(2,2,1,1,0,0)}\alpha_1^2$ .

**Type xv ( $\mathbf{D}_5$ ).** Type  $xv$ , which has already been considered in [BB], has the following properties:

- It is given by the the following quadrics in  $\mathbb{P}^4$ :

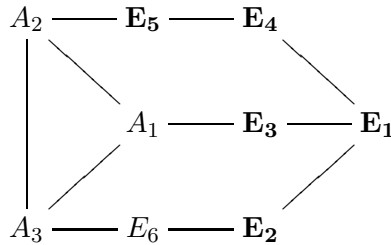
$$x_0x_1 - x_2^2 = x_3^2 + x_0x_4 + x_1x_2 = 0$$

- The singularity  $p = (0 : 0 : 0 : 0 : 1)$  gives five exceptional divisor  $E_1, \dots, E_5$ , and its line is  $E_6 = \{x_0 = x_2 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_6$ , and  $-K_{\tilde{S}} = (6, 5, 3, 4, 2, 4)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_6, \alpha_1, \alpha_2, \alpha_3] / (\eta_3\alpha_1^2 + \eta_2\eta_6^2\alpha_3 + \eta_4\eta_5^2\alpha_2^3),$$

where the degree of the relation is  $(6, 6, 3, 4, 2, 6)$ , and  $\deg(\alpha_1) = A_1 = (3, 3, 1, 2, 1, 3) = \psi^*(\{a^3 : ab^2 : a^2b : 0 : -a^3\})$  gives the projection  $\psi_1$  from  $E_6$  with  $\alpha_1 = \psi_1^*(x_3)$ ,  $\deg(\alpha_2) = A_2 = (2, 2, 1, 1, 0, 2) = \psi^*(\{a^2 : 0 : 0 : ab : -b^2\})$  gives the projection  $\psi_2 : x \mapsto (x_0 : x_2)$  with  $\alpha_2 = \psi_2^*(x_2)$ , and  $\deg(\alpha_3) = A_3 = -K_{\tilde{S}} = \psi^*(\{-a^4 : -b^4 : a^2b^2 : ab^3 : 0\})$  gives the anticanonical embedding  $\phi$  with  $\alpha_3 = \phi^*(x_4)$ .

- The extended Dynkin diagram is:



where  $A_1, A_2, A_3$  intersect in  $(1 : 0 : 0 : 0 : 0)$ .

- For the anticanonical embedding,

$$(\phi^*(x_i)) = (\eta^{(6,5,3,4,2,4)}\alpha_2, \eta^{(2,1,1,2,2,0)}\alpha_2^2, \eta^{(4,3,2,3,2,2)}\alpha_2, \eta^{(3,2,2,2,1,1)}\alpha_1, \alpha_3).$$

## 8. DEGREE 3

The classification of cubic Del Pezzo surfaces is classical and goes back to L. Schläfli (1863). Together with their number of lines, the list in Table 5 can be found in [BW79].

type	singularities	number of lines	type
0	—	27	$\geq 2$ relations
<i>i</i>	$\mathbf{A}_1$	21	$\geq 2$ relations
<i>ii</i>	$2\mathbf{A}_1$	16	$\geq 2$ relations
<i>iii</i>	$\mathbf{A}_2$	15	$\geq 2$ relations
<i>iv</i>	$3\mathbf{A}_1$	12	$\geq 2$ relations
<i>v</i>	$\mathbf{A}_2 + \mathbf{A}_1$	11	$\geq 2$ relations
<i>vi</i>	$\mathbf{A}_3$	10	$\geq 2$ relations
<i>vii</i>	$4\mathbf{A}_1$	9	$\geq 2$ relations
<i>viii</i>	$\mathbf{A}_2 + 2\mathbf{A}_1$	8	$\geq 2$ relations
<i>ix</i>	$\mathbf{A}_3 + \mathbf{A}_1$	7	$\geq 2$ relations
<i>x</i>	$2\mathbf{A}_2$	7	$\geq 2$ relations
<i>xi</i>	$\mathbf{A}_4$	6	$\geq 2$ relations
<i>xii</i>	$\mathbf{D}_4$	6	1 relation
<i>xiii</i>	$\mathbf{A}_3 + 2\mathbf{A}_1$	5	1 relation
<i>xiv</i>	$2\mathbf{A}_2 + \mathbf{A}_1$	5	1 relation
<i>xv</i>	$\mathbf{A}_4 + \mathbf{A}_1$	4	1 relation
<i>xvi</i>	$\mathbf{A}_5$	3	$\geq 2$ relations
<i>xvii</i>	$\mathbf{D}_5$	3	1 relation
<i>xviii</i>	$3\mathbf{A}_2$	3	toric
<i>xix</i>	$\mathbf{A}_5 + \mathbf{A}_1$	2	1 relation
<i>xx</i>	$\mathbf{E}_6$	1	1 relation

TABLE 5. Del Pezzo surfaces of degree 3

**Proposition 13.** *The cubic Del Pezzo surfaces of Table 5 belong to the following groups:*

- *Type xviii is toric, with cyclic Dynkin diagram*

$$(-2, -2, -1, -2, -2, -1, -2, -2, -1).$$

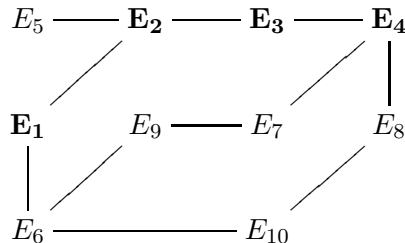
- *The Cox rings of types xii, xiii, xiv, xv, xvii, xix, xx have 10 generators and one relation.*
- *The Cox rings of types 0, i, ii, iii, iv, v, vi, vii, viii, ix, x, xi, xvi have at least 11 generators and at least two independent relations.*

*Proof.* As at the beginning of the proof of Lemma 12, it is not hard to see that in order to obtain a toric cubic surface, the only possibility is to blow up a certain point on type *x* of degree 4. Therefore, type *xviii* is the only toric cubic Del Pezzo surface. It has been studied extensively, for example in [Bre98], [Fou98], [HBM99].

Using Lemma 2 and Lemma 4, and since the number of negative curves is at least 11, types 0 to *x* must have more than one relation. For types

*xi, xvi*, this is not as obvious because the number of negative curves is 10 and 8, respectively.

For type  $xi$ , the negative curves give 10 necessary generators of the Cox ring. The extended Dynkin diagram of the exceptional divisors  $E_1, \dots, E_4$  and the six lines  $E_5, \dots, E_{10}$  is:



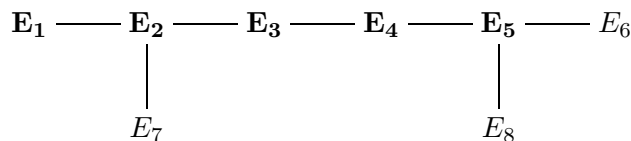
A basis of  $\text{Pic}(\widetilde{S})$  is given by  $E_1, \dots, E_7$ , and in terms of this basis,

$$\begin{aligned} E_8 &= (1, 1, 0, -1, 1, 1, -1), & E_9 &= (1, 2, 1, 0, 2, 0, -1), \\ E_{10} &= (0, 1, 1, 1, 1, -1, 1), & -K_{\tilde{\Sigma}} &= (2, 3, 2, 1, 2, 1, 0). \end{aligned}$$

The divisors  $E_1, \dots, E_{10}$  generate the effective cone. We check that  $A = (1, 1, 1, 1, 0, 1, 1)$  is nef. Therefore, we calculate  $\dim \Gamma(A) = \chi(A) = 2$  by Riemann-Roch. However, the subring generated by non-zero sections of  $\Gamma(E_i)$  does not give two linearly independent sections in  $\Gamma(A)$ . Therefore,  $\text{Cox}(\tilde{S})$  must have more than 10 generators.

For type *xvi*, we have the following information:

- The singularity gives the exceptional divisors  $E_1, \dots, E_5$ , and there are three lines  $E_6, E_7, E_8$ .
- The extended Dynkin diagram of negative curves is:



- A basis of  $\text{Pic}(\widetilde{S})$  is  $E_1, \dots, E_7$ , with  $E_8 = (1, 2, 1, 0, -1, -1, 2)$  and  $-K_{\widetilde{S}} = (2, 4, 3, 2, 1, 0, 3)$ .
- Three of the 13 generators of the nef cone are  $B_1 = (0, 1, 1, 1, 1, 1)$ ,  $B_2 = (1, 3, 2, 1, 0, -1, 3)$ ,  $B_3 = (1, 2, 2, 2, 2, 2, 1)$ , with  $\dim(\Gamma(B_1)) = \dim(\Gamma(B_2)) = 2$  and  $\dim(\Gamma(B_3)) = 3$ .

As for type *viii* of degree 4, we check that a minimal system of generator of  $\text{Cox}(\tilde{S})$  can be assumed to contain  $\eta_1, \dots, \eta_8$  with non-zero  $\eta_j \in \Gamma(E_j)$ , and  $\beta_1, \beta_2$  of degree  $B_1, B_2$ , respectively. However, it is not hard to check that the subring of  $\text{Cox}(\tilde{S})$  generated by  $\eta_1, \dots, \eta_8, \beta_1, \beta_2$  contains only two linearly independent sections in degree  $B_3$ . Therefore,  $\text{Cox}(\tilde{S})$  must have more than 10 generators.

For each type whose Cox ring has exactly 10 generators, the calculations follow below.  $\square$

*Remark 14.* By [BW79], some types of cubic surfaces do not have a single normal form, but a family with one or more parameters. More precisely, this happens exactly for the types  $0, i, ii, iii, iv, v, vi, x$ . Furthermore, by [BW79,

Lemma 4], the  $\mathbf{D}_4$  cubic surface (type  $xii$ ) is the only surface which has more than one normal form, but not a family.

The two different surfaces with a  $\mathbf{D}_4$  singularity are also discussed in [HT04, Remark 4.1].

**Type xii ( $\mathbf{D}_4$ ).** As mentioned above, type  $xii$  has two different forms:

- The first is given by

$$x_0(x_1 + x_2 + x_3)^2 - x_1x_2x_3 = 0,$$

the second by

$$x_0(x_1 + x_2 + x_3)^2 + x_1x_2(x_1 + x_2) = 0.$$

- The singularity is  $p = (1 : 0 : 0 : 0)$ , giving four exceptional divisor  $E_1, \dots, E_4$ . In the first form, the six lines are  $E_5 = \{x_1 = x_2 + x_3 = 0\}$ ,  $E_6 = \{x_2 = x_1 + x_3 = 0\}$ ,  $E_7 = \{x_3 = x_1 + x_2 = 0\}$ ,  $E_8 = \{x_0 = x_1 = 0\}$ ,  $E_9 = \{x_0 = x_2 = 0\}$ ,  $E_{10} = \{x_0 = x_3 = 0\}$ . In the second form,  $E_{10}$  is replaced by  $\{x_0 = x_1 + x_2 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is given by  $E_1, \dots, E_7$ , with

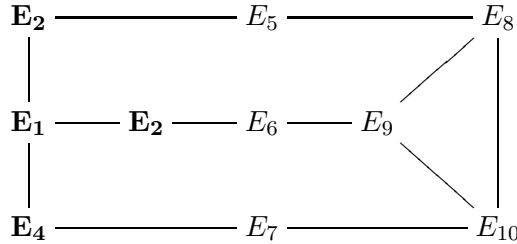
$$\begin{aligned} E_8 &= (1, 0, 1, 1, -1, 1, 1), & E_9 &= (1, 1, 0, 1, 1, -1, 1), \\ E_{10} &= (1, 1, 1, 0, 1, 1, -1), & -K_{\tilde{S}} &= (3, 2, 2, 2, 1, 1, 1). \end{aligned}$$

- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_{10}] / (\eta_2\eta_5^2\eta_8 + \eta_3\eta_6^2\eta_9 + \eta_4\eta_7^2\eta_{10} - A \cdot \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7),$$

where the constant  $A$  is 1 for the first form and 0 for the second form, and the degree of the relation is  $(1, 1, 1, 1, 1, 1, 1)$ .

- The extended Dynkin diagram is:



The lines  $E_8, E_9, E_{10}$  meet in one point only in case of the second form.

- For the first form, the anticanonical embedding  $\phi : S \rightarrow \mathbb{P}^3$  is given by

$$(\phi^*(x_i)) = (\eta_8\eta_9\eta_{10}, \eta^{(2,2,1,1,2,0,0)}\eta_8, \eta^{(2,1,2,1,0,2,0)}\eta_9, \eta^{(2,1,1,2,0,0,2)}\eta_{10}),$$

and furthermore,  $\phi^*(x_1 + x_2 + x_3) = \eta^{(3,2,2,2,1,1,1)}$ . The second form differs from this in the following way:  $\phi^*(-x_1 - x_2) = \eta^{(2,1,1,2,0,0,2)}\eta_{10}$ , and  $\phi^*(x_3) = \eta^{(3,2,2,2,1,1,1)} + \eta^{(2,1,1,2,0,0,2)}\eta_{10}$ .

**Type xiii ( $\mathbf{A}_3 + 2\mathbf{A}_1$ ).** Type  $xiii$  has the following properties:

- It is given by the following cubic in  $\mathbb{P}^3$ :

$$x_3^2(x_1 + x_2) + x_0x_1x_2 = 0.$$

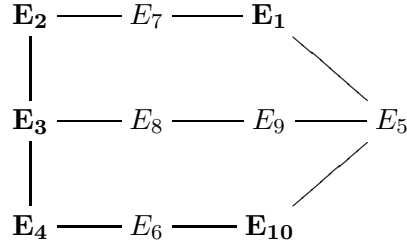


- The singularities in  $p_1 = (0 : 1 : 0 : 0)$  and  $p_2 = (0 : 0 : 1 : 0)$  give exceptional divisors  $E_1, E_{10}$ , respectively, and the singularity in  $p_3 = (1 : 0 : 0 : 0)$  gives  $E_2, E_3, E_4$ . The five lines are given by  $E_5 = \{x_0 = x_3 = 0\}$ ,  $E_6 = \{x_1 = x_3 = 0\}$ ,  $E_7 = \{x_2 = x_3 = 0\}$ ,  $E_8 = \{x_1 = x_2 = 0\}$ ,  $E_9 = \{x_0 = x_1 + x_2 = 0\}$ .
- A basis of the  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , and  $E_8 = (1, 0, -1, -1, 1, -1, 1)$ ,  $E_9 = (0, 1, 1, 1, -1, 1, 1)$ ,  $E_{10} = (1, 1, 0, -1, 0, -2, 2)$ , and  $-K_{\tilde{S}} = (2, 2, 1, 0, 1, -1, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_{10}] / (\eta_4 \eta_6^2 \eta_{10} + \eta_1 \eta_2 \eta_7^2 + \eta_8 \eta_9),$$

where the relation is of degree  $(1, 1, 0, 0, 0, 0, 2)$ .

- The extended Dynkin diagram is:



- For the anticanonical embedding,

$$\begin{aligned}
 (\phi^*(x_i)) = & (\eta^{(1,0,0,0,2,0,0)} \eta_9 \eta_{10}, \eta^{(0,1,2,2,0,2,0)} \eta_8 \eta_{10}, \\
 & \eta^{(1,2,2,1,0,0,2)} \eta_8, \eta^{(1,1,1,1,1,1,1)} \eta_{10}),
 \end{aligned}$$

and furthermore,  $\phi^*(-x_1 - x_2) = \eta^{(0,1,2,1,0,0,0)} \eta_8^2 \eta_9$ .

**Type xiv ( $2\mathbf{A}_2 + \mathbf{A}_1$ ).** Type *xiv* has the following properties:

- Its equation is

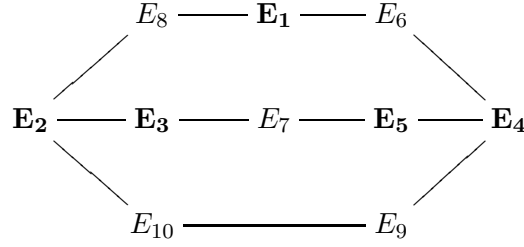
$$x_3^2(x_1 + x_3) + x_0 x_1 x_2 = 0.$$

- The singularity  $(0 : 1 : 0 : 0)$  gives an exceptional divisor  $E_1$ , and  $(1 : 0 : 0 : 0)$  gives  $E_2, E_3$ , and  $(0 : 0 : 1 : 0)$  gives  $E_4, E_5$ . The lines are  $E_6 = \{x_0 = x_3 = 0\}$ ,  $E_7 = \{x_1 = x_3 = 0\}$ ,  $E_8 = \{x_2 = x_3 = 0\}$ ,  $E_9 = \{x_0 = x_1 + x_3 = 0\}$ ,  $E_{10} = \{x_2 = x_1 + x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , and  $E_8 = (-1, 0, 1, 0, 1, -1, 2)$ ,  $E_9 = (-1, 1, 2, -1, 1, -2, 3)$ ,  $E_{10} = (1, -1, -1, 1, 0, 2, -1)$ , and  $-K_{\tilde{S}} = (0, 1, 2, 1, 2, 0, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_{10}] / (\eta_3 \eta_5 \eta_7^2 + \eta_1 \eta_6 \eta_8 + \eta_9 \eta_{10}),$$

where the relation is of degree  $(0, 0, 1, 0, 1, 0, 2)$ .

- The extended Dynkin diagram is:



- For the anticanonical embedding,

$$(\phi^*(x_i)) = (\eta^{(1,0,0,2,1,2,0)}\eta_9, \eta^{(0,1,2,1,2,0,3)}\eta_9, \eta^{(1,2,1,0,0,0,0)}\eta_8^2\eta_{10}, \eta^{(1,1,1,1,1,1,1)}\eta_8),$$

$$\text{and furthermore, } \phi^*(-x_1 - x_3) = \eta^{(0,1,1,1,1,0,1)}\eta_9\eta_{10}.$$

**Type xv ( $\mathbf{A}_4 + \mathbf{A}_1$ ).** Type  $xv$  has the following properties:

- Its equation is

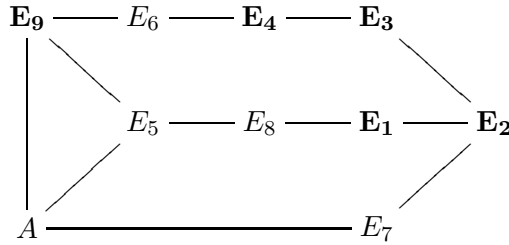
$$x_2x_3^2 + x_1^2x_3 + x_0x_1x_2 = 0.$$

- The singularity  $(0 : 0 : 1 : 0)$  gives an exceptional divisor  $E_9$ , and  $(1 : 0 : 0 : 0)$  gives  $E_1, \dots, E_4$ . The lines are  $E_5 = \{x_0 = x_3 = 0\}$ ,  $E_6 = \{x_1 = x_3 = 0\}$ ,  $E_7 = \{x_1 = x_2 = 0\}$ ,  $E_8 = \{x_2 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , and  $E_8 = (0, 1, 1, 1, -1, 1, 1)$ ,  $E_9 = (1, 2, 1, 0, -1, -1, 2)$ , and  $-K_{\tilde{S}} = (2, 4, 3, 2, -1, 1, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_9, \alpha] / (\eta_1\eta_5\eta_8^2 + \eta_3\eta_4^2\eta_6^3\eta_9 + \eta_7\alpha),$$

where the relation is of degree  $(1, 2, 2, 2, -1, 2, 2)$ , and  $\deg(\alpha) = A = (1, 2, 2, 2, -1, 2, 1) = \psi^*(\{(0 : ab : a^2 : -b^2)\})$  describes the projection  $\psi_1$  from  $E_5$ , with  $\alpha = \psi_1^*(x_0)$ .

- The extended Dynkin diagram is:



Here,  $A, E_5, E_9$  meet in one point.

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta_5\eta_9\alpha, \eta^{(1,2,2,2,0,2,1)}\eta_9, \eta^{(2,3,2,1,0,0,2)}\eta_8, \eta^{(1,1,1,1,1,1,0)}\eta_8\eta_9).$$

**Type xvii ( $\mathbf{D}_5$ ).** Type  $xvii$  has the following properties:

- Its equation is

$$x_3x_0^2 + x_0x_2^2 + x_1^2x_2 = 0.$$

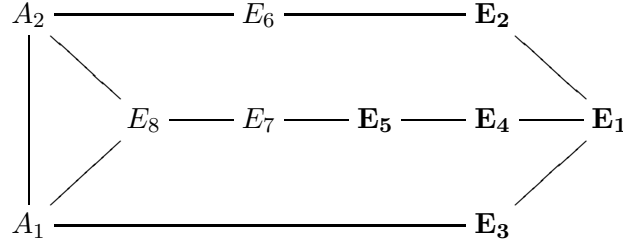
- The singularity  $(0 : 0 : 0 : 1)$  gives exceptional divisors  $E_1, \dots, E_5$ . The lines are  $E_6 = \{x_0 = x_1 = 0\}$ ,  $E_7 = \{x_0 = x_2 = 0\}$ , and  $E_8 = \{x_2 = x_3 = 0\}$ .

- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , with  $E_8 = (2, 2, 1, 1, 0, 2, -1)$  and  $-K_{\tilde{S}} = (4, 3, 2, 3, 2, 2, 1)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_8, \alpha_1, \alpha_2] / (\eta_2 \eta_6^2 \alpha_2 + \eta_4 \eta_5^2 \eta_7^3 \eta_8 + \eta_3 \alpha_1^2),$$

where the relation has degree  $(2, 2, 1, 2, 2, 2, 2)$ , and  $\deg(\alpha_1) = A_1 = (1, 1, 0, 1, 1, 1, 1) = \psi^*(\{(a^2 : 0 : ab : -b^2)\})$  describes a projection  $\psi_1$  from  $E_6$  with  $\alpha_1 = \psi_1^*(x_1)$ , and  $\deg(\alpha_2) = A_2 = (2, 1, 1, 2, 2, 0, 2) = \psi^*(\{(a^2 : ab : -b^2 : 0)\})$  describes the projection  $\psi_2$  from  $E_8$  with  $\alpha_2 = \psi_2^*(x_3)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, E_8$  meet in  $(1 : 0 : 0 : 0)$ .

- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta^{(4,3,2,3,2,2,1)}, \eta^{(3,2,2,2,1,1,0)} \alpha_1, \eta^{(2,1,1,2,2,0,2)} \eta_8, \eta_8 \alpha_2).$$

**Type xix ( $A_5 + A_1$ ).** Type *xix* has the following properties:

- Its equation is

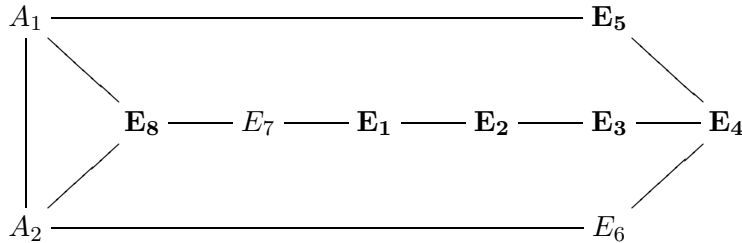
$$x_1^3 + x_2 x_3^2 + x_0 x_1 x_2 = 0.$$

- The singularity  $(0 : 0 : 1 : 0)$  gives an exceptional divisor  $E_8$ , and  $(1 : 0 : 0 : 0)$  gives  $E_1, \dots, E_5$ . The lines are  $E_6 = \{x_1 = x_2 = 0\}$  and  $E_7 = \{x_1 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , with  $E_8 = (-1, 0, 1, 2, 1, 2, -2)$  and  $-K_{\tilde{S}} = (1, 2, 3, 4, 2, 3, 0)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_8, \alpha_1, \alpha_2] / (\eta_1^3 \eta_2^2 \eta_3 \eta_7^4 \eta_8 + \eta_5 \alpha_1^2 + \eta_6 \alpha_2),$$

where the relation is of degree  $(2, 2, 2, 2, 1, 2, 2)$ , and  $\deg(\alpha_1) = A_1 = (1, 1, 1, 1, 0, 1, 1) = \psi^*(\{(a^2 : ab : -b^2 : 0)\})$  (which describes the projection  $\psi_1$  from  $E_7$ , with  $\alpha_1 = \psi_1^*(x_3)$ ), and  $\deg(\alpha_2) = A_2 = (2, 2, 2, 2, 1, 1, 2) = \psi^*(\{(0 : ab^2 : -a^3 : b^3)\})$  (which describes the projection  $\psi_2$  from  $(0 : 0 : 1 : 0)$ , with  $\alpha_2 = \psi_2^*(x_0)$ ).

- The Dynkin diagram is:



- The anticanonical embedding is given by

$$(\phi^*(x_i)) = (\eta_8 \alpha_2, \eta^{(2,2,2,2,1,1,2)} \eta_8, \eta^{(1,2,3,4,2,3,0)} \eta^{(1,1,1,1,1,0,1)} \eta_8 \alpha_1).$$

**Type xx ( $\mathbf{E}_6$ ).** Type  $xx$  has been considered in [HT04, Section 3]. Its properties are:

- The equation is

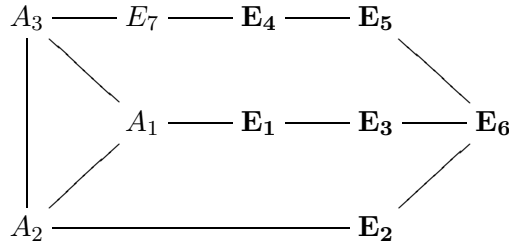
$$x_1 x_2^2 + x_2 x_0^2 + x_3^3 = 0.$$

- The singularity  $(0 : 1 : 0 : 0)$  gives exceptional divisors  $E_1, \dots, E_6$ , and there is a unique line  $E_7 = \{x_2 = x_3 = 0\}$ .
- A basis of  $\text{Pic}(\tilde{S})$  is  $E_1, \dots, E_7$ , with  $-K_{\tilde{S}} = (2, 3, 4, 4, 5, 6, 3)$ .
- The Cox ring is

$$\text{Cox}(\tilde{S}) = \mathbb{K}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2, \alpha_3] / (\eta_4^2 \eta_5 \eta_7^3 \alpha_3 + \eta_2 \alpha_2^2 + \eta_1^2 \eta_3 \alpha_1^3),$$

where the relation is of degree  $(2, 3, 4, 6, 6, 6, 6)$ , and  $\deg(\alpha_1) = A_1 = (0, 1, 1, 2, 2, 2, 2) = \psi^* (\{(ab : a^2 : -b^2 : 0)\})$  describes a projection  $\psi_1$  from  $E_7$  with  $\alpha_1 = \psi_1^*(x_3)$ , and  $\deg(\alpha_2) = A_2 = (1, 1, 2, 3, 3, 3, 3) = \psi^* (\{(0 : -a^3 : b^3 : ab^2)\})$  describes the projection  $\psi_2$  from the singularity with  $\alpha_2 = \psi_2^*(x_0)$ , and  $\deg(\alpha_3) = A_3 = (2, 3, 4, 4, 5, 6, 3) = -K_{\tilde{S}} = \psi^* (\{(a^3 : 0 : -b^3 : a^2 b)\})$  describes the anticanonical embedding  $\phi$  with  $\alpha_3 = \phi^*(x_1)$ .

- The extended Dynkin diagram is:



Here,  $A_1, A_2, A_3$  meet in  $(0 : 0 : 1 : 0)$ .

- The anticanonical embedding  $\phi$  is given by

$$(\phi^*(x_i)) = (\eta^{(1,2,2,1,2,3,0)} \alpha_2, \alpha_3, \eta^{(2,3,4,4,5,6,3)} \eta^{(2,2,3,2,3,4,1)} \alpha_1).$$

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